Example 5: **Compare Two Exponential Functions**

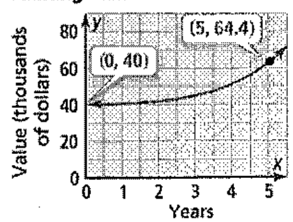
A museum purchased a painting and a sculpture in the same year. Their changing values are modeled as shown. Find the average rate of change of the value of each art work over the 5-year period. Which art work's value is increasing more quickly?

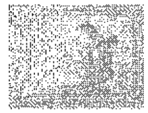
Sculpture Value

$f(x) = 50(1.075)^x$

(value in thousands of dollars in x years)

Painting Value





Rate of Change: $\frac{\Delta Y}{\Delta X}$

$f(0) = 50(1.075)^0 = 50$

$f(5) = 50(1.075)^5 = 71.78$

$\frac{71.78 - 50}{5 - 0} = \frac{21.78}{5} = 4.356$

$\frac{64.4 - 40}{5 - 0} = \frac{24.4}{5} = 4.88$

greater rate of change.

ETP: Why is the slope of the line containing two points on the graph of an exponential function considered the "average" rate of change?

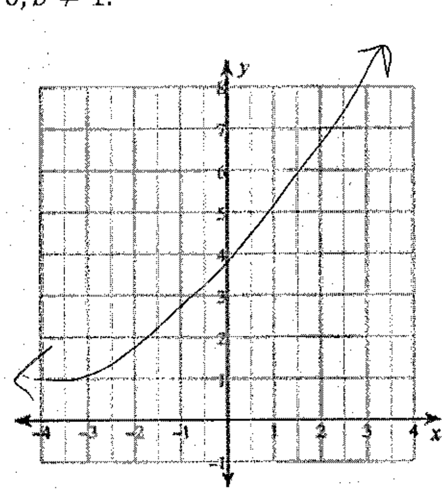
- How else could you compare the two functions besides comparing their average rate of change over 5-yr period?

6-1 Key Features of Exponential Functions

Target: I can recognize the key features of exponential functions.

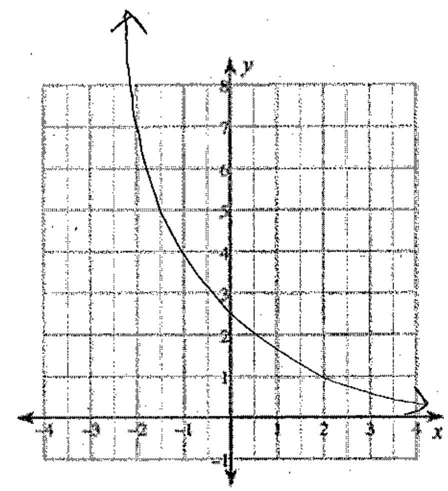
Essential Question: How do graphs and equations reveal key features of exponential growth and decay functions?

An ___exponential___ function is any function of the form $y = a \cdot b^x$ where a and b are constants with $a \neq 0$, and $b > 0, b \neq 1$.



Growth

$b > 1$



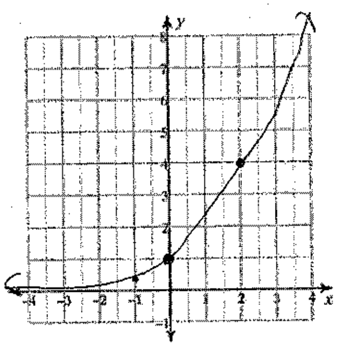
Decay

$b < 1$

Example 1: **Identify Key Features of Exponential Functions**

What are the key features of each function? Include domain, range, intercepts, asymptotes, and end behavior.

A. $f(x) = 2^x$



Domain: $\mathbb{R}$

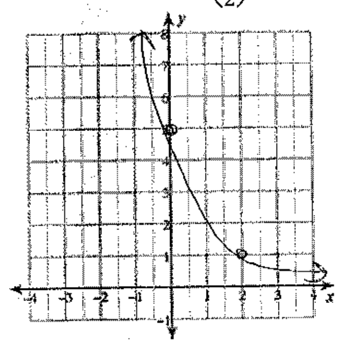
Range: $y > 0$ or $(0, \infty)$

Y-int: 1

End behavior: $x \to -\infty,\ y \to 0$
$x \to \infty,\ y \to \infty$

Asymptote: $y = 0$

B. $g(x) = 5\left(\frac{1}{2}\right)^x$



Domain: $\mathbb{R}$

Range: $y > 0$ or $(0, \infty)$

Asymptote $y = 0$

Y-int: 5

End behavior:
$x \to -\infty,\ y \to \infty$
$x \to \infty,\ y \to 0$

ETP: Consider the functions $f(x) = 2^x$ and $g(x) = 5\left(\frac{1}{2}\right)^x$.
What information tells you whether the functions are increasing or decreasing?

CONCEPT Exponential Growth and Decay Models

Exponential growth and **exponential decay** functions model quantities that increase or decrease by a fixed percent during each time period. Given an initial amount a and the rate of increase or decrease r, the amount $A(t)$ after t time periods is given by:

Exponential Growth Model	Exponential Decay Model
$A(t) = a(1 + r)^t$	$A(t) = a(1 - r)^t$
$a > 0, b > 1, b = 1 + r$	$a > 0, 0 < b < 1, b = 1 - r$

The **growth or decay factor** is equal to b, and is the ratio between two consecutive y-values.

Example 4: **Interpret and Exponential Function**

A car was purchased for \$24,000. The function $y = 24 \cdot 0.8^x$ can be used to model the value of the car (in thousands of dollars) x years after it was purchased.

A. Does the function represent exponential growth or decay?

$b < 1$, so decay

B. What is the rate of decay for this function? What does it mean?

$b = 1 - r = 1 - .8 = .2 = 20\%$

means decrease by 20% each year.

C. Graph the function on a reasonable domain. What do the y-intercept and asymptote represent? When will the value of the car be about \$5,000?

Quadrant 1

24 = y-intercept: starting value
asymptote: $y = 0$, value of car after x years
when $y = 5$, $x \approx 7$

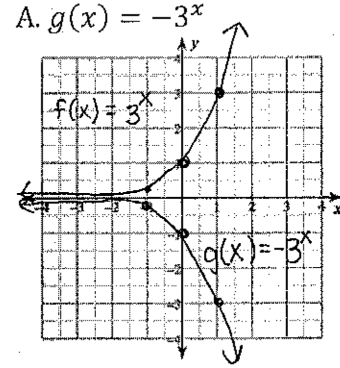
So after 7 years

ETP: Why is $0 < x < 10$ a reasonable domain?
The value of another car can be modeled by $y = 31 \cdot .77^x$. Compare to ex.

Example 2: Graph transformations of Exponential Functions

Graph each function. Describe the graph in terms of transformations of the parent function $f(x) = 3^x$. How do the asymptote and intercept of the given function compare to the asymptote and intercept of the parent function?

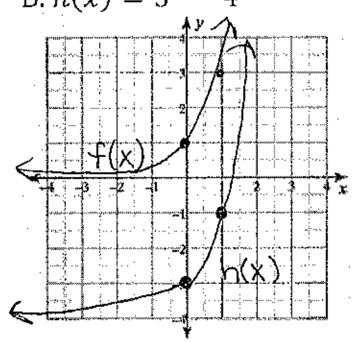
A. $g(x) = -3^x$



compare: same asymptote

contrast: reflects over x-axis
$g(x) = -f(x)$
intercept is negative

B. $h(x) = 3^x - 4$

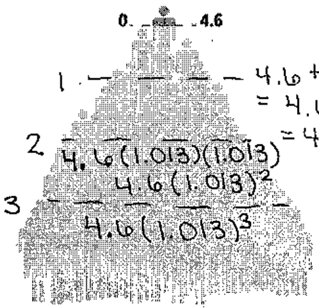


compare: increasing
same shape (no stretch)

constrast: down 4
$h(x) = f(x) - 4$
asymptote shifted down 4
($y = 0$ to $y = -4$)

Example 3: **Model with Exponential Functions**

t (years since 2010) **P (population in millions)**



1 — $4.6 + 4.6(.013)$
$= 4.6(1+.013)$
$= 4.6(1.013)$

2 $4.6(1.013)(1.013)$
$4.6(1.013)^2$

3 $4.6(1.013)^3$

The population of a large city was about 4.6 million in the year 2010 and grew at a rate of 1.3% for the next four years.

A. What exponential function models the population of the city over that 4-year period? $P = 4.6(1.013)^t$ → years since 2010 → growth factor ↑initial population

P = population after t years

B. If the population continues to grow at the same rate, what will the population be in 2040?

↳ $t = 40$

$P = 4.6(1.013)^{40}$

$= 6.78$ million people

ETP: What is the relationship between numbers on the left side of graphic and expression on right side.

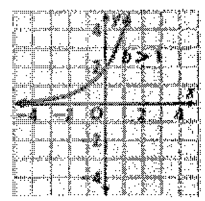
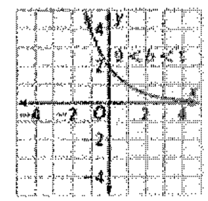
Why is growth factor 1.013 rather than .013?

Example 6: Error Analysis

Charles claimed the function $f(x) = \left(\frac{3}{2}\right)^x$ represents exponential decay. Explain the error Charles made.

$\frac{3}{2} > 1$, so it is growth

CONCEPT SUMMARY **Key Features of Exponential Functions**

	Exponential Growth	Exponential Decay
GRAPHS	 Growth factor: $1 + r$	 Decay factor: $1 - r$
EQUATIONS	$y = a \cdot b^x$, for $b > 1$	$y = a \cdot b^x$, for $0 < b < 1$
KEY FEATURES	Domain: All real numbers Range: $\{y \mid y \geq 0\}$ Intercepts: $(0, a)$ Asymptote: x-axis	Domain: All real numbers Range: $\{y \mid y \geq 0\}$ Intercepts: $(0, a)$ Asymptote: x-axis
END BEHAVIOR	As $x \to -\infty$, $y \to 0$ As $x \to \infty$, $y \to \infty$	As $x \to -\infty$, $y \to \infty$ As $x \to \infty$, $y \to 0$
MODELS	Growth: $A(t) = a(1 + r)^t$	Decay: $A(t) = a(1 - r)^t$

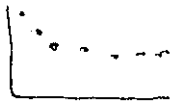
ETP: Consider $y = 500(.86)^x$. How can you find the rate of decay?

0	212°F
2	180°F
3.5	170°F
5	162°F
7.5	145°F
10	138°F
15	125°
20	117°

Example 6 **Use Regression to Find an Exponential Model**

Randy is making soup. The soup reaches the boiling point and then, as shown by the data, begins to cool off. Randy wants to serve the soup when it is about 80°F, or about 10 degrees above room temperature (68°).

*picture added to student copy



Data is not linear. Once it approaches room temp, it will no longer cool

A. Explain why the temperature might follow an exponential decay curve as it approaches room temperature.

ETP: Why can't the soup cool to below 68°F

B. Find an exponential model for the data. Use your model to determine when Randy should serve the soup.

To add list:
Stat
edit

L_1	L_2

To find exp reg:
stat → calc
option 0 ExpReg

ETP: What does 0.9492 mean in the given context
- Can you use a model that does not involve adding 68 to the exponential function?

6-2 Exponential Models

Target: I can write exponential models in different ways to solve problems.

Essential Question: How can you develop exponential models to represent and interpret situations?

Example 1 **Rewrite an Exponential Function to Identify a Rate**

In 2015, the population of a small town was 8,000. The population is increasing at a rate of 2.5% per year. Rewrite an exponential growth function to find the monthly grown rate.

Initial population: 8,000
Growth Rate: .025

$$y = 8000(1+.025)^t$$ ← years

For months: 12t would = 1 year

$$y = 8000(1.025^{1/12})^{12t}$$

$$y \approx 8000(1.00206)^{12t}$$

Monthly increase ≈ .206%

Struggling: How does monthly rate compare to the annual rate

CONCEPT Compound Interest

When interest is paid monthly, the interest earned after the first month becomes part of the new principal for the second month, and so on. Interest is earned on interest already earned. This is compound interest.

The compound interest formula is an exponential model that is used to calculate the value of an investment when interest is compounded.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

P = the initial principal invested

r = annual interest rate, written as a decimal

n = number of compounding periods per year

A = the value of the account after t years

Example 2 **Understand Compound Interest**

Tamira invests $5,000 in an account that pays 4% annual interest. How much will there be in the account after 3 years if the interest is compounded annually, semi-annually, quarterly, or monthly?

Annually $(n=1)$: $A = 5000\left(1+\frac{.04}{1}\right)^{1(3)} = 5{,}624.32$

Semi-Annually $(n=2)$ $A = 5000\left(1+\frac{.04}{2}\right)^{2(3)} = 5{,}630.81$

Quarterly $(n=4)$ $A = 5000\left(1+\frac{.04}{4}\right)^{4(3)} = 5{,}634.13$

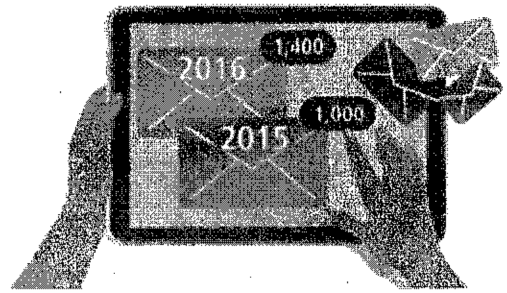
Monthly $(n=12)$ $A = 5000\left(1+\frac{.04}{12}\right)^{4(12)} = 5{,}636.36$

ETP: • Why is 0.04 used in the formula for r rather than 4?

• When banks advertise an interest rate of 3% compounded quarterly, do they really pay 3% every quarter?

Example 5 **Use Two Points to Find and Exponential Model**

Tia knew that the number of e-mails she sent was growing exponentially. She generated a record of the number of e-mails she sent each year since 2009. What is an exponential model that describes the data?



Growth factor

$\frac{1400}{1000} = 1.4$

↑

so b

$y = a \cdot b^x$

$\frac{14}{1.4^7} = \frac{a(1.4)^7}{1.4^7}$

$1.33 \approx a$

↑ in hundreds

So $y = 1.33(1.4)^x$

ETP: • Why is 7 substituted for x?

• How can you find the growth factor when the data points have x-values that are 2 units apart?

Example 3 **Understanding Continuously Compounded Interest**

Consider an investment of \$1 in an account that pays a 100% annual interest rate for one year. The equation $A = 1\left(1+\frac{1}{n}\right)^{n(1)} = \left(1+\frac{1}{n}\right)^{n}$ gives the amount in the account after one year for the number of compounding periods n. Find the value of the account for the number of periods given in the table.

Number of Periods, n	
1	$\left(1+\frac{1}{1}\right)^{1} = 2$
10	$\left(1+\frac{1}{10}\right)^{10} = 2.593742...$
100	$\left(1+\frac{1}{100}\right)^{100} = 2.70481....$
1,000	$\left(1+\frac{1}{1000}\right)^{1000} = 2.7169239$
10,000	$\left(1+\frac{1}{10000}\right)^{10000} = 2.7181459...$
100,000	$\left(1+\frac{1}{100000}\right)^{10000} = 2.71826...$

This number is called natural base

ETP: · Are the values of $(1+\frac{1}{n})^n$ exact values

· Is it necessary to calculate the value of the account to nine decimals?

The natural base e is defined as the value that the expression $\left(1+\frac{1}{x}\right)^x$ approaches as $x \to +\infty$. The number e is an irrational number.

$e =$ 2.718281828459...

The number e is the base in the continuously compounded interest formula.

Formula: $A = Pe^{rt}$ ← exponent

$P =$ initial principle

$e =$ the natural base

$r =$ annual interest rate (decimal!)

$A =$ value after t years

t = time

Example 4 **Find Continuously Compounded Interest**

Regina invests \$12,600in an account that earns 3.2% annual interest, compounded continuously. What is the value of the account after 12 years? Round your answer to the nearest dollar.

$P = 12{,}600 \quad r = .0032 \quad t = 12$

$A = Pe^{rt}$

$A = 12600e^{.0032(12)}$

$\approx 18{,}498.63$

The value of the account after 12 years is \$18,499.

ETP • What part of the continuously compounded interest formula represents the concept of continuous?

CONCEPT SUMMARY Writing Exponential Models

	General Exponential Model	Compound Interest	Continuously Compounded Interest
ALGEBRA	$y = a \cdot b^x$	$A = P\left(1+\frac{r}{n}\right)^{nt}$	$A = Pe^{rt}$
NUMBERS	A necklace costs \$250 and increases in value by 2% per year. $a =$ initial amount \$250 $b =$ growth factor 1.02 $x =$ number of years $y = 250(1.02)^x$	A principal of \$3,000 is invested at 5% annual interest, compounded monthly, for 4 years. $P = 3{,}000$ $r = 5\%$ $n = 12$ compounding periods per year $t = 4$ years $A = 3000\left(1+\frac{0.05}{12}\right)^{(12)(4)}$	A principal of \$3,000 is invested at 5% continuously compounded interest for 4 years. $P = 3{,}000$ $r = 5\%$ $t = 4$ years $A = 3000e^{(0.05)(4)}$

ETP: Write models for the balance in an account with \$500 principal and an interest rate of 4.5% compounded monthly and continuously.

How does the value of n affect the value of A?

6-3 Logarithms

I can evaluate and simplify logarithms.

Essential Question: What are logarithms and how are they evaluated?

Example 1 Understand Logarithms

Solve the equations $2x = 8$ and $2^x = 8$

inverse→ $\frac{2x}{2} = \frac{8}{2}$ $x = 3$

$x = 4$

"To what exponent would you raise the base 2 to get 8?"

↑the inverse is called a logarithm

The logarithm base b of x is defined as follows:

$\log_b x = y$ if and only if $b^y = x$, for $b > 0, b \neq 1$, and $x > 0$.

The logarithmic function $y = \log_b x$ is the inverse of the exponential function $y = b^x$.

CONCEPT Exponential and Logarithmic Forms

Exponential form shows that a base raised to an exponent equals the result.

$$a^b = c$$

Logarithmic form shows that the log of the result with the given base equals the exponent.

$$\log_a c = b$$

When written in logarithmic form, the number that was the result of the exponential equation is often called the argument.

Example 1

ETP: How is solving $2^x = 8$ similar to solving $2x = 8$? different?

You can calculate the value of x in $2^x = 8$ without rewriting the function as $\log_2 8 = x$. When might it be necessary to use this notation.

Example 2 Convert Between Exponential and Logarithmic Forms

A. What is the logarithmic form of $3^4 = 81$? $\log_3 81 = 4$ "log base 3 of 81 is 4."

B. What is the exponential form of $\log_{10} 1{,}000 = 3$? $10^3 = 1000$

called common log
typically do not write 10.

ETP: Why might you find it necessary to convert forms?

Example 3 Evaluate Logarithms

What is the value of each logarithmic expression?

ETP Must all parts of a logarithmic expression be positive?

A. $\log_5 125 = 3$ because $5^3 = 125$	B. $\log_{\frac{1}{4}} 16 = -2$ $\frac{1}{4}^{?} = 16$
C. $\log_0 3$ $3^{?} = 0$ There is none So UNDEFINED	D. $\log_2 2^8 = 8$ $2^{?} = 2^{8}$ same

CONCEPT Exponential and Logarithmic Forms

Exponential form shows that a base raised to an exponent equals the result.

$$a^b = c$$

Logarithmic form shows that the log of the result with the given base equals the exponent.

$$\log_a c = b$$

When written in logarithmic form, the number that was the result of the exponential equation is often called the argument.

Example 4 Evaluate Common and Natural Logarithms

What is the value of each logarithmic expression to the nearest ten-thousandth?

A. $\log 900$

≈ 2.9542

B. $\ln e$

$\log_e e$
cancels so
1

C. $\ln(-1.87)$

cannot be negative
So
Undefined

ETP: Why can the exponential form of the expression be used to check your answer?

Example 5 Solve Equations With Logarithms

What is the solution to each equation? Round to the nearest thousandth.

A. $25 = 10^{x-1}$

$\log_{10} 25 = x - 1$

$\log_{10}(25) + 1 = x$

ETP: How is solving exponential and logarithmic equations similar to solving linear? How is it different?

B. $\ln(2x + 3) = 4$

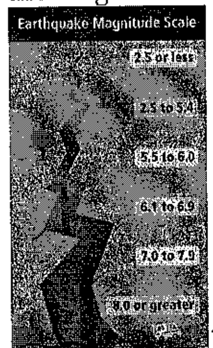
$e^4 = 2x + 3$

$\frac{e^4 - 3}{2} = \frac{2x}{2}$

$25.799 \approx x$

Example 6 Use Logarithms to Solve Problems

The seismic energy, x, in joules can be estimated based on the magnitude, m, of an earthquake by the formula $x = 10^{1.5m+12}$. What is the magnitude of an earthquake with a seismic energy of 4.2×10^{20}?



$x = 10^{1.5m+12}$

$4.2 \times 10^{20} = 10^{1.5m+12}$

$\log(4.2 \times 10^{20}) = 1.5m + 12$

$\frac{\log(4.2 \times 10^{20}) - 12}{1.5} = \frac{1.5m}{1.5}$

$5.75 \approx m$

ETP: How does converting forms of the equation help to find the magnitude of the earthquake.

The magnitude of earthquake is about 5.75

CONCEPT SUMMARY Logarithms

	Exponential Form	Logarithmic Form
ALGEBRA	$b^x = y$	$\log_b y = x$
WORDS	The base raised to the exponent is equal to a result.	The logarithm with a base b of the result (or argument) is equal to the exponent.
NUMBERS	$3^4 = 81$	$\log_3 81 = 4$

ETP: Explain the relationship between logarithms & exponents. When is it useful to convert between exponential and logarithmic forms?

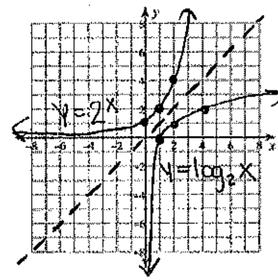
6-4 Logarithmic Functions

I can graph logarithmic functions and find equations of the inverses of exponential and logarithmic functions.

Essential Question: How is the relationship between logarithmic and exponential functions revealed in the key features of their graph?

Example 1 Identify Key Features of Logarithmic Functions

Graph $y = \log_2 x$. What are the domain, range, x-intercept, and asymptote? What is the end behavior of the graph?



Domain: $(0, \infty)$ or $x > 0$

Range: $(-\infty, \infty)$ or $\mathbb{R}$

x-intercept: 1

Asymptote: $x = 0$ (y-axis)

End behavior: As $x \to 0$, $y \to -\infty$; $x \to \infty$, $y \to \infty$

*notice its inverse is $y = 2^x$ so Domain and range switched

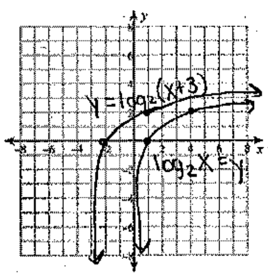
ETP: when graphing $y = \log_2 x$, why is it helpful to start w/ $y = 2^x$?

Example 2 Graph Transformations of Logarithmic Functions

Graph the function. How do you know the asymptote and x-intercept of the given function compare to those of the parent function?

$$g(x) = \log_2(x + 3)$$

left 3



vertical asymptote shifted left 3.

$f(x+3)$ ← another way to write a shift

ETP: How is graphing transformations of logarithmic functions the same as graphing transformations of exponential functions? Different?

$$g(x) = \pm(\log \pm x - h) + k$$

reflection over x-axis if −

reflection over y-axis if −

− right, + left

+ up, − down

Example 3 **Inverses of Exponential and Logarithmic Functions**

What is the equation of the inverse of the functions?

A. $f(x) = 10^{x+1}$

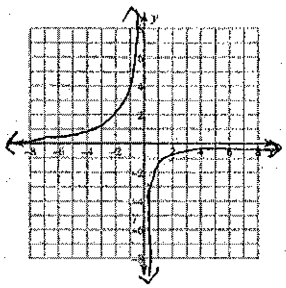
① switch x and y
② get y by itself

$x = 10^{y+1}$

$\log_{10} x = y+1$

$\log(x) - 1 = y$

$f^{-1}(x) = \log(x) - 1$



*$\log_{10}$ is known as common log so 10 is not needed.

B. $g(x) = \log_7(x + 5)$

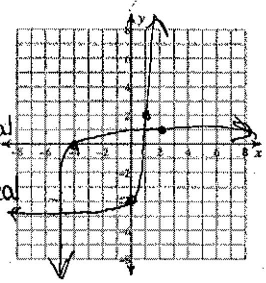
$x = \log_7(y+5)$

$7^x = y + 5$

$7^x - 5 = y$

$g^{-1}(x) = 7^x - 5$

ETP: Why does a horizontal shift in an exponential equation become a vertical shift in its inverse?



Example 4 **Interpret the Inverse of a Formula Involving Logarithms**

A company uses this function to relate sales revenue R and advertising costs, a:

$$R = 12\log(a + 1) + 25$$

What is the equation of the inverse of the formula? Which equation would be easier to use to find a value of a for a particular value of R?

$R = 12\log(a+1) + 25$

$\frac{R-25}{12} = \frac{12\log(a+1)}{12}$

$\frac{R-25}{12} = \log_{10}(a+1)$

$10^{\frac{R-25}{12}} = a+1$

$10^{\frac{R-25}{12}} - 1 = a$

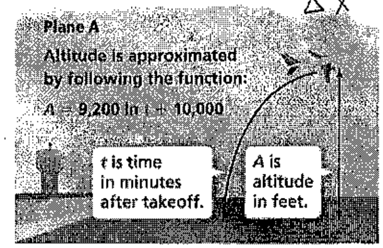
ETP: Why did you need to subtract 25 and divide by 12 before rewriting the equation in exponential form?

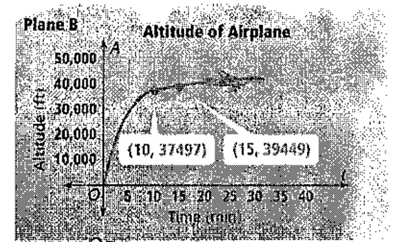
The inverse since R is known and we're finding a.

ETP: How is finding the average rate of change on an interval similar to finding rate of change, or slope, of the line between two points?

Example 5 **Compare Two Logarithmic Functions**

Logarithmic functions can approximate the altitude of a plan over time. Which plane's altitude shows the greater rate of change over the interval $10 \le t \le 15$? $\frac{\Delta y}{\Delta x}$





$f(10) = 9200\ln(10) + 10000 = 34914$

$f(15) = 9200\ln(15) + 10000 = 31{,}184$

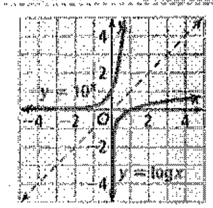
$\frac{34914 - 31184}{15-10} = \frac{3730}{5} = 746\text{ft/min}$

$\frac{39449 - 37497}{15-10} = \frac{1952}{5} = 390\text{ft/min}$

Average Rate of Change of altitude of Plane A is > than Plane B

CONCEPT SUMMARY **Logarithmic Functions**

GRAPH



The functions are inverses so their graphs are reflections of each other across the line with equation $y = x$.

EQUATIONS	$y = \log x$	$y = 10^x$
KEY FEATURES	Domain: $\{x \mid x > 0\}$ Range: all real numbers x-intercept: 1 Asymptote: y-axis	Domain: all real numbers Range: $\{y \mid y > 0\}$ y-intercept: 1 Asymptote: x-axis
END BEHAVIOR	As $x \to 0$, $y \to -\infty$ As $x \to \infty$, $y \to \infty$	As $x \to -\infty$, $y \to 0$ As $x \to \infty$, $y \to \infty$

ETP: How are the domain and range of $y = \log x$ related to domain and range of $y = 10^x$?

How are the x-int. and asymptote of $y = \log x$ affected if the graph is translated 3 units to the right?

6-5 Properties of Logarithms

I can use properties of logarithms to rewrite expressions.

Essential Question: How are the properties of logarithms used to simplify expressions and solve logarithmic equations?

CONCEPT Properties of Logarithms

For positive numbers b, m, and n with $b \neq 1$, the following properties hold.

$\log_b mn = \log_b m + \log_b n$ .... Product Property of Logarithms

$\log_b \frac{m}{n} = \log_b m - \log_b n$ .... Quotient Property of Logarithms

$\log_b m^n = n \log_b m$ .... Power Property of Logarithms

ETP: What do you notice about product property of Logarithms?

Example 1 Prove a Property of Logarithm

How can you prove the Product Property of Logarithms?

Let $x = \log_b m$ and $y = \log_b n$. Then $b^x = m$ and $b^y = m$.

$b^x b^y = m \cdot n$

$b^{x+y} = mn$

$\log_b mn = x + y$

$\log_b mn = \log_b m + \log_b n$

ETP: Why would the equations $x = \log_b m$ and $y = \log_b n$ be used to start the proof?

Example 2 Expand Logarithmic Expressions

How can you use the properties of logarithms to expand each expression?

A. $\log_5(a^2 b^7)$

$\log_5 a^2 + \log_5 b^7$

$2\log_5 a + 7\log_5 b$

ETP: Why might it be useful to know how to expand logarithmic expressions?

B. $\ln(\frac{25}{3})$

$\ln 25 - \ln 3$

Example 3 Write Expressions as Single Logarithms

What is each expression written as a single logarithm?

A. $4\log_4 m + 3\log_4 n - \log_4 p$

$\log_4 m^4 + \log_4 n^3 - \log_4 p$

$\log_4 \frac{m^4 n^3}{p}$

B. $3\ln 2 - 2\ln 5$

$\ln 2^3 - \ln 5^2$

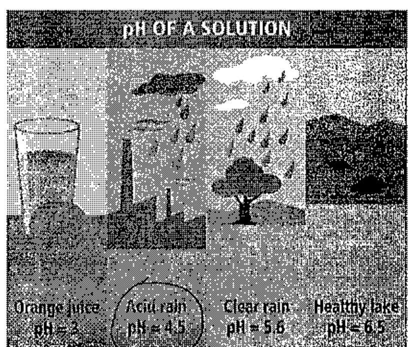
$\ln \frac{2^3}{5^2}$

$\ln$

ETP: How do the processes of expanding a logarithm and writing an expression as a single logarithm relate to each other?

Example 4 Apply Properties of Logarithms

The pH of a solution is a measure of its concentration of hydrogen ions. This concentration (measured in moles per liter) is written $[H^+]$ and is given by the formula $pH = \log\frac{1}{[H^+]}$. What is the concentration of hydrogen ions in the acid rainfall?



$4.5 = \log\frac{1}{[H^+]}$

$4.5 = \log 1 - \log[H^+]$

$4.5 - \underbrace{\log 1}_{=0} = -\log[H^+]$

$-4.5 = \log[H^+]$

$10^{-4.5} = [H^+]$

≈ 0.0000316

ETP: What does the + in H^+ represent? Does it affect how you solve the equation?

Example 5 Evaluate Logarithmic Expressions by Changing the Base

How can you use base 10 logarithms to evaluate base 2 logarithms?

Ex. $\log_2 3$ is equal to:

$\frac{\log 3}{\log 2}$

*Good to know if you have a basic scientific calculator

$\boxed{\log_b m = \frac{\log_a m}{\log_a b}}$

ETP: What does each variable represent when you use the Power Property of logarithms?

This illustrates the <u>Change of Base Formula</u>.

Example 6 Use the Change of Base Formula

What is the solution of the equation $2^x = 7$? Express the solution as a logarithm and then evaluate. Round to the nearest thousandth.

$2^x = 7$

$\log_2 7 = x$

$\frac{\log 7}{\log 2} = x$

$x \approx 2.807$

Check: $2^{2.807} \approx 7$

ETP: Why can you solve the equation using both base 10 logarithms and natural logarithms?

CONCEPT SUMMARY Properties of Logarithms

	Product Property	Quotient Property	Power Property	Change of Base
ALGEBRA	$\log_b(mn) = \log_b m + \log_b n$	$\log_b(\frac{m}{n}) = \log_b m - \log_b n$	$\log_b(m^n) = n \cdot \log_b m$	$\log_b m = \frac{\log_a m}{\log_a b}$
WORDS	The log of a product is the sum of the logs.	The log of a quotient is the difference of the logs.	The log of a number raised to a power is the power multiplied by the log of the number.	The log base b of a number is equal to the log base a of the number divided by the log base a of b.
NUMBERS	$\log_2(20) = \log_2(4) + \log_2(5)$	$\log_{10}(\frac{2}{3}) = \log_{10}2 - \log_{10}3$	$\log_3(16) = 4 \cdot \log_3 2$	$\log_5 7 = \frac{\log 7}{\log 5}$

ETP: How are the properties of logarithms useful when solving problems?

Example 6 **Solve Logarithmic and Exponential Equations by Graphing**

What is the solution to $\underbrace{\log(2x+1)^5}_{Y_1} = \underbrace{x-2}_{Y_2}$?

Second
trace
intersect
enter x≈3

* 2 intersections, so 2 solutions!

-.329 and 8.204

ETP: Could you let Y1 = X-2 and Y2 = 5log(2X+1)? Explain.

6-6 Exponential and Logarithmic Equations

I can solve exponential and logarithmic equations

Essential Question: How do properties of exponents and logarithms help you solve equations?

CONCEPT Property of Equality for Exponential Equations

Symbols	Suppose $b > 0$ and $b \neq 1$, then $b^x = b^y$ if and only if $x = y$.
Words	If two powers of the same base are equal, then their exponents are equal; if two exponents are equal, then the powers with the same base are equal.

An exponential equation is an equation that contains variables in the exponents.

Example 1 **Solve Exponential Equations Using a Common Base**

What is the solution to $\left(\frac{1}{2}\right)^{x+7} = 4^{3x}$?

ETP: Why was each side of the equation written with an equivalent expression using 2 as the base?
• Could the expression have been rewritten by using a common base of ½, 4, or 10 rather than 2?

$$(2^{-1})^{x+7} = (2^2)^{3x}$$
$$2^{-x-7} = 2^{6x}$$
$$-x-7 = 6x$$
$$+x \qquad +x$$
$$-7 = 7x$$
$$-1 = x$$

Example 2 **Rewrite Exponential Equations Using Logarithms**

How can you rewrite the equation $17 = 4^x$ using logarithms?

$$\log_4 17 = x$$
$$\frac{\log 17}{\log 4} = x$$
$$= x$$

OR

$$10^{\log 17} = 10^{\log 4^x}$$
$$\log 17 = \log 4^x$$

ETP: Why is $10^{(\log 17)}$ equivalent to 17?

A logarithmic equation contains one or more logarithms of variable expressions.

Example 5 **Solve Logarithmic Equations**

What is the solution to $\ln(x^2 - 16) = \ln(6x)$?

$$x^2 - 16 = 6x$$
$$x^2 - 6x - 16 = 0$$
$$(x-8)(x+2) = 0$$
$$x = 8 \qquad x = -2$$

Check:

$$\ln(8^2 - 16) = \ln(6(8))$$
$$\ln(48) = \ln(48)$$

$$\ln((-2)^2 - 16) = \ln(6(-2))$$
$$\ln(-12) = \ln(-12)$$

Undefined

ETP: What is an extraneous solution and how do they occur?

CONCEPT Property of Equality for Logarithmic Equations

Symbols	If $x > 0$, then $\log_b x = \log_b y$ if and only if $x = y$.
Words	If two logarithms (exponents) of the same base are equal, then the quantities are equal; if two quantities are equal, and the bases are the same, then the logarithms (exponents) are equal.

Example 3 Solve Exponential Equations Using Logarithms

What is the solution to $3^{x+1} = 5^x$?

$$\log 3^{x+1} = \log 5^x$$

$$(x+1)\log 3 = x \log 5$$

$$x \log 3 + \log 3 = x \log 5$$

$$x \log 3 - x \log 5 = -\log 3$$

$$\frac{x(\log 3 - \log 5)}{\log 3 - \log 5} = \frac{-\log 3}{\log 3 - \log 5}$$

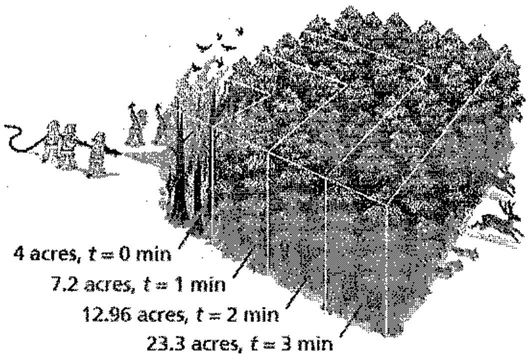
$$x = \frac{-\log 3}{\log 3 - \log 5}$$

$$x \approx 2.15$$

ETP: Why are log 3 and x log 5 subtracted from both sides of the equation?

Example 4 Use an Exponential Model

The diagram shows how a forest fire grows over time. The fire department can contain a 160-acre fire without needing additional resources. About how many minutes does it take for a fire to become too big for the fire department to contain without additional resources? Round to the nearest minute.



$\frac{7.2}{4} = 1.8 = r$

start

$\frac{160}{4} = \frac{4(1.8)^t}{4}$

$40 = 1.8^t$

$\log_{1.8} 40 = t$

$6.276 \approx t$

The fire department has a little more than 6 minutes to contain the fire before they will require additional resources.

ETP: How do you know that the ratios of the number of acres are all 1.8?
Why does knowing the ratios are all 1.8 translate to knowing that the growth rate b is also 1.8?

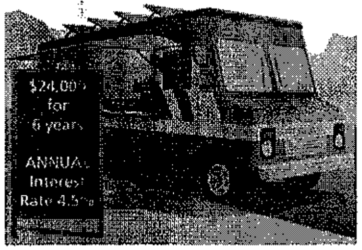
CONCEPT SUMMARY Exponential and Logarithmic Equations

	Property of Equality for Exponential Equations		Property of Equality for Logarithmic Equations	
ALGEBRA	If b is a positive number other than 1, $b^x = b^y$ if and only if $x = y$.		If b is a positive number other than 1, $\log_b x = \log_b y$ if and only if $x = y$.	
WORDS	If two powers of the same base are equal, then their exponents are equal.	If two exponents are equal, then the powers with the same base are equal.	If two logarithms of the same base are equal, then the arguments are equal.	If two arguments are equal and the bases are the same, then the logarithms are equal.
NUMBERS	If $2^x = 2^4$, then $x = 4$.	If $x = 4$, then $2^x = 2^4$.	If $\log_3 x = \log_3 8$, then $x = 8$.	If $x = 8$, then $\log_3 x = \log_3 8$.

ETP: Give an example of how using the Property of Equality for Logarithmic Equations can help simplify the process of solving a logarithmic equation.
· How can you solve an exponential equation by graphing?

Example 6 **Use a Finite Geometric Series**

Isabel wants to borrow \$24,000 for 6 years with an annual interest rate of 4.5% to purchase a share in a food truck business. What will be her monthly payment?



$$A = \frac{P}{\sum_{k=1}^{n}\left(\frac{1}{1+i}\right)^k}$$

A = monthly amount
P = principal
n = # of months
i = interest rate

$$A = \frac{24000}{\sum_{k=1}^{72}\left(\frac{1}{1+.00375}\right)^k}$$

$$= \frac{24000}{62.417}$$

$$\approx 384.51$$

Isabel's monthly payment would be about \$384.51

ETP: How do you use the values 6 years and 4.5% annual interest rate to solve the problem?
How does understanding finite geometric series help you solve this problem?

6-7 Geometric Sequences and Series

I can identify, write, and use geometric sequences and series.

Essential Question: How can you represent and use geometric sequences and series?

A <u>geometric sequence</u> sequence is a sequence with a constant ratio between consecutive terms. This ratio is called the <u>common ratio</u>, *r*.

Example 1 **Identify Geometric Sequences**

A. Is the sequence shown in the table a geometric sequence? If so, write a recursive definition for the sequence. ETP: How are geometric sequences similar to arithmetic sequences? How are the different? Suppose the sixth term of the sequence is 964. Is the sequence still geometric?

Term Number (n)	Term (a_n)
1	4
2	12
3	36
4	108
5	324

$\times 3$

So $r=3$

*could also do $a_2 \div a_1$, $a_3 \div a_2$

Recursive: $a_n = \begin{cases} 4, & n=1 \\ 3\cdot(a_{n-1}), & n>1 \end{cases}$

$\boxed{\text{General Formula } a_n = \begin{cases} a_1, & n=1 \\ a_{n-1}(r), & n>1 \end{cases}}$

B. Is the sequence 12, 9.6, 7.68, 6.144, ... a geometric sequence? If so, write the recursive definition for the sequence.

$\frac{9.6}{12} = .8$ $\quad \frac{7.68}{9.6} = .8$ $\quad \frac{6.144}{7.68} = .8$ $\quad$ so yes, $r = .8$

$$a_n = \begin{cases} 12, & n=1 \\ .8a_{n-1}, & n>1 \end{cases}$$

Example 2 Translate Between Recursive and Explicit Definitions

A. Given the recursive definition $a_n = \begin{cases} 5, & n = 1 \\ a_{n-1}\left(\frac{1}{2}\right), & n > 1 \end{cases}$, what is the explicit definition for the geometric sequence?

$5, 2.5, 1.25, \ldots$

$a_n = 5\left(\frac{1}{2}\right)^{n-1}$

ETP: When the terms of a geometric sequence have a negative common ratio, why do the signs of consecutive terms alternate?

B. Given the explicit definition $a_n = 3(2)^{n-1}$, what is the recursive definition for the geometric sequence?

$a_n = \begin{cases} 3, & n = 1 \\ 2a_{n-1}, & n > 1 \end{cases}$

ETP: Why is the power of the common ratio in the explicit definition of a geometric sequence $n-1$ instead of n?

- What information do the recursive and explicit definitions have in common?

Example 5 Number of Terms in a Finite Geometric Series

A. How many terms are in the geometric series $200 + 300 + 450 + \cdots + 7{,}688.7$?

$a_n = a_1 \cdot r^{n-1}$

$r = \frac{300}{200} = \frac{3}{2} = 1.5$

$\frac{7688.7}{200} = \frac{200(1.5)^{n-1}}{200}$

$38.4435 = 1.5^{n-1}$

$\log_{1.5} 38.4435 = n-1$

$1 + \log_{1.5} 38.4435 = n$

$10 \approx n$

10 terms in the geometric series

ETP: Why do you need to use logarithm when solving this equation?

B. The sum of a geometric series 11,718. The first term of the series is 3, and its common ratio is 5. How many terms are in the series?

$S_n = \frac{a_1(1-r^n)}{1-r}$

$11{,}718 = \frac{3(1-5^n)}{1-5}$

$-4 \cdot 11{,}718 = \frac{3(1-5^n)}{-4} \cdot -4$

$\frac{-46{,}872}{3} = \frac{3(1-5^n)}{3}$

$-15{,}624 = 1-5^n$

$-15{,}625 = -5^n$

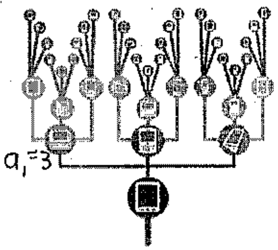
$15{,}625 = 5^n$

$\log_5 15625 = n$

$6 = n$

Example 3 **Solve Problems With Geometric Sequences**

A phone tree is when one person calls a certain number of people, then those people each call the same number of people, and so on. In the fifth round of calls, 243 people were called.



$a_1 = 3$

A. Write an explicit definition to find the number of people called in each round.

$a_1 = 3 \qquad a_5 = 243$

$a_5 = a_1 r^{n-1}$

$243 = 3r^{5-1}$

$\frac{243}{3} = \frac{3r^4}{3}$

$(81)^{1/4} = (r^4)^{1/4}$

$3 = r$

$\boxed{a_n = 3(3)^{n-1}}$

ETP: To write the explicit formula for a geometric sequence, you need to know the value of 1st term and common ratio. Which is given? Which do you need to calculate from the given info?

B. How many people were called in the eighth round of the phone tree?

$a_8 = 3(3)^{8-1}$

$a_8 = 6{,}561$

ETP: How can you use the explicit formula for the geometric sequence to determine how many people were called in the eighth round?

A geometric ___series___ is the sum of the terms of a geometric sequence. S_n represents the sum of a geometric sequence with n terms.

Example 4 **Formula for the Sum of a Finite Geometric Series**

A. How can you find the sum of a finite geometric series?

$$S_n = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \ldots + a_1 r^{n-1}$$

$$\boxed{S_n = \frac{a_1(1-r^n)}{1-r}}$$

*multiply EVERY term by r

$$rS_n = a_1 r + a_1 r^2 + a_1 r^3 + a_1 r^4 + \ldots + a_1 r^n$$

$$S_n - rS_n = a_1 - a_1 r^n$$

$$\frac{S_n(1-r)}{1-r} = \frac{a_1(1-r^n)}{1-r}$$

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

ETP: Why do you multiply by r in the second equation when finding the sum of a finite geometric series?

B. Write the expanded form of the series. What is the sum?

$$\sum_{n=1}^{7} 3\left(\frac{2}{3}\right)^{n-1}$$

$$= 3\left(\tfrac{2}{3}\right)^{1-1} + 3\left(\tfrac{2}{3}\right)^{2-1} + 3\left(\tfrac{2}{3}\right)^{3-1} + 3\left(\tfrac{2}{3}\right)^{4-1} + 3\left(\tfrac{2}{3}\right)^{5-1} + 3\left(\tfrac{2}{3}\right)^{6-1} + 3\left(\tfrac{2}{3}\right)^{7-1}$$

$$= 3 + 2 + \frac{4}{3} + \frac{8}{9} + \frac{16}{27} + \frac{32}{81} + \frac{64}{243}$$

$$= \frac{2{,}059}{243}$$

OR

$$S_7 = \frac{3(1-\frac{2}{3}^7)}{1-\frac{2}{3}}$$

$$= \frac{2{,}049}{243}$$

To check in calculator:
Alpha window, 2

In a geometric sequence, the ratio defined by a term divided by the previous term is a constant, r. Alternately, any term in a geometric sequence multiplied by r gives the next term.

The sequence 1, 5, 25, 125, 625, ... is a geometric sequence, since $r = 5$.

WORDS	ALGEBRA	EXAMPLE
Each term in the sequence is r times the previous term.	The recursive definition for a geometric sequence is $a_n = \begin{cases} a_1, & n = 1 \\ a_{n-1} \cdot r, & n > 1 \end{cases}$	$a_n = \begin{cases} 1 & n = 1 \\ 5a_{n-1}, & n > 1 \end{cases}$
The fourth term in a sequence is the first term multiplied by three common ratios.	The explicit definition is $a_n = a_1 r^{n-1}$	$a_n = 1(5)^{n-1}$
You can find the sum of a certain number of terms in a geometric series.	For a finite geometric series with $r \neq 1$ $\sum_{m=1}^{n} a_1 r^{m-1} = \frac{a_1(1-r^n)}{(1-r)}$	The sum of the first five terms is $\frac{1(1-5^5)}{1-5} = \frac{-3{,}124}{-4} = 781$

ETP: How does the explicit definition of a geometric sequence compare to an exponential function?